Hopf algebras in SupLat and set-theoretical YBE solutions

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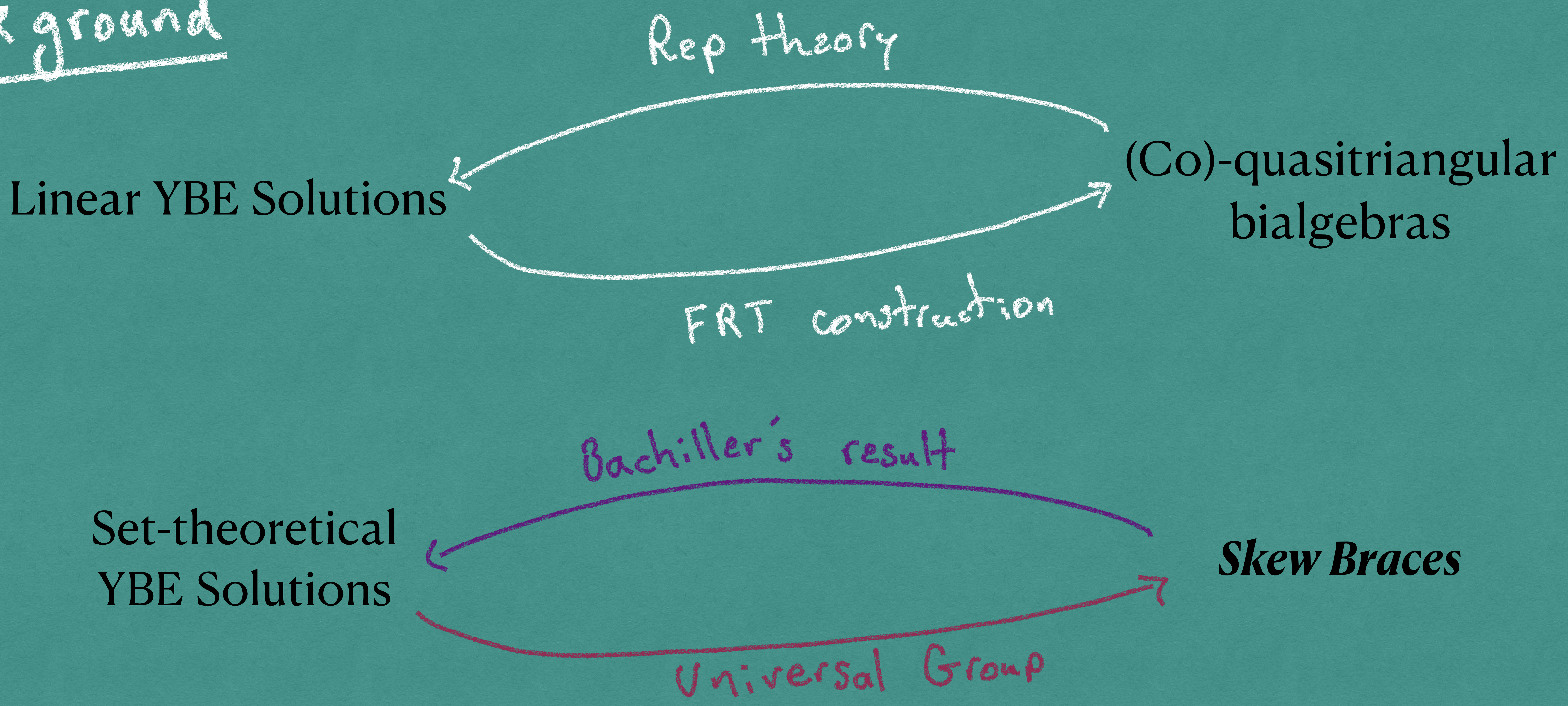
Based on: arXiv:2001.08673 [G1], arXiv:2005.07183 [G2]

Plan

- Background/Motivation
- Set-theoretical YBE solutions, Skew braces
- The category
- Main Results
- Application 1: Transmutation
- Application 2: Categorical FRT
- Application 3: Drinfeld Twists
- Outlook

background



Question: Can skew braces be viewed as Hopf algebras in a reasonable category?

background

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a good categorical interpretation should:

- Explain aspects of skew braces in terms of Hopf algebras
- Categorical FRT should recover the universal skew brace
- Allow us to apply Hopf algebra techniques to obtain new skew braces (Help Classification)

Set-theoretical YBE

A set X + a map $r : X \times X \to X \times X$ satisfying

$$r_{23}r_{12}r_{23} = r_{12}r_{23}r_{12}$$

is called a **set-theoretical YBE solution**.

- Notation $r(x, y) = (\sigma_x(y), \gamma_y(x))$, for $\sigma_x, \gamma_y : X \to X$
- If r is bijective: $r^{-1}(x, y) = (\tau_x(y), \rho_y(x))$
- Graphical notation:
- Solution is **non-degenerate** if σ_x, γ_y are bijections for all $x, y \in X$.
- Solution is **involutive** if $r^2 = \mathrm{id}_{X\times X}$

Groups with braiding Operators

[LYZ] A **braiding operator** on a group (G, m, e) is a map $r : G \times G \to G \times G$ satisfying

$$r(e,g) = (g,e), \quad r(g,e) = (e,g)$$

$$rm_{12} = m_{23}r_{12}r_{23}$$

$$rm_{23} = m_{12}r_{23}r_{12}$$

$$mr = m$$

It follows that r has to satisfy YBE, and is invertible and non-degenerate!

Universal Group of solution (X, r) ⇝ $G(X,r) = \langle x \in X \mid x \,.\, y = \sigma_x(y) \,.\, \gamma_y(x),\ \forall x, y \in X \rangle$

Skew Brace

[GV] A **skew (left) brace** consists of a set B + two group structures $(B, .)$ and $(B, \star)$

$$a\,.\,(b \star c) = (a\,.\,b) \star a^{\star} \star (a\,.\,c)$$

where a^{-1} and $a^{\star}$ = multiplicative inverses of a with respect to . and $\star$

!Notation Warning! Authors (usually) use $\circ$ and . instead of . and $\star$

- $(B, \star)$ called additive group of skew brace
- If $(B, \star)$ is abelian then we have a ***brace***

Skew Braces

$(G, m, \star)$, where

$$x \star y := x \,.\, \sigma_x^{-1}(y)$$

Skew braces

$(B, ., \star)$

Groups with braiding operators

$(G, m, e) + r$

$(B, ., e)$

$$r(a, b) = \left(a^\star \star (a \,.\, b), (a^\star \star (a \,.\, b))^{-1} \,.\, a \,.\, b\right)$$

SupLat

Main Theorems in [G1]

From any Hopf algebra H in SupLat, we can construct a group called its ***remnant*** $\mathsf{R}(H)$

Any co-quasitriangular structure on H gives a braiding operator on $\mathsf{R}(H)$

Any skew brace can be recovered in this way!

SupLat

Reference: Joyal-Tierney: An extension of the Galois theory of Grothendieck

- Objects: partially ordered sets $(\mathscr{L}, \leq)$, where any subset $S \subseteq \mathscr{L}$, has a least upper bound, $\bigvee S$, called ***joins***
- Morphisms: join-preserving maps

Notation: $\vee_{i\in I}\, a_i$ for $\vee \{a_i \mid i \in I\}$

- All objects in SupLat have ***meets***: (they're complete lattices!)

$$\wedge S = \vee \{a \mid a \leq s \ , \forall s \in S\}$$

- Free Lattices: For any set X, its power-set $\mathscr{P}(X)$, with $\vee = \cup$ and?? is a complete lattice.
- Fun Example: the set of positive integers, $\mathrm{div}(z)$, which divide a positive integer $z \in \mathbb{N}$

SupLat

Reference: Joyal-Tierney: An extension of the Galois theory of Grothendieck

- Notation: Top element of $\mathscr{L} = \vee \mathscr{L}$, Bottom element of $\mathscr{L} = \varnothing$
- SupLat is complete and co-complete
- SupLat is symmetric monoidal closed:

$$\mathscr{M} \otimes \mathscr{N} = \text{Quotient of } \mathscr{P}(\mathscr{M} \times \mathscr{N}) \text{ by relations}$$

$$\{(\vee_{i\in I} m_i, n)\} = \cup_{i\in I} \{(m_i, n)\}$$

$$\{(m, \vee_{i\in I} n_i)\} = \cup_{i\in I} \{(m, n_i)\}$$

- Monoidal unit: $\{\varnothing, 1\}$

SupLat

Lemma. [G1] Dualisable objects in SupLat are free lattices.

We have a faithful monoidal functor $\mathcal{P}(-): \mathbf{Sets} \xrightarrow{\otimes} \mathbf{SupLat}$ (factoring as Sets → Rel → SupLat)

Fact. An invertible morphism $r : \mathscr{P}(X) \to \mathscr{P}(Y)$ must be of the form $\mathscr{P}(b)$ for a bijection $b : X \to Y$.

invertible set-theoretical YBE solutions $\xleftrightarrow{1:1}$ invertible YBE solutions in SupLat on dualizable objects

Hopf algebras in SupLat

Any distributive complete lattice $\mathscr{L}$

$$\vee_{i\in I}(a_i \wedge b) = (\vee_{i\in I} a_i) \wedge b \text{ and } \vee_{i\in I}(b \wedge a_i) = b \wedge (\vee_{i\in I} a_i)$$

has a natural bialgebra structure with $\vee\mathscr{L}$ acting as unit, $m(a,b) = a \wedge b$, $\epsilon(\vee\mathscr{L}) = 1$, $\epsilon(\text{other elements}) = \varnothing$ and $\Delta(a) = \{(a,1)\} \vee \{(1,a)\}$.

Main Theorems in [G1]

From any Hopf algebra H in SupLat, We can construct a group called its ***remnant*** $\mathsf{R}(H)$

Main Theorems in [G1]

Any co-quasitriangular structure on H gives a braiding operator on $\mathsf{R}(H)$

A co-quasitriangular structure consists of $\mathscr{R} : H \otimes H \to \mathscr{P}(1)$

$$\mathscr{R}(a\,.\,b, c) = \mathscr{R}(b, c_{(1)})\,.\,\mathscr{R}(a, c_{(2)})$$

$$\mathscr{R}(a, b\,.\,c) = \mathscr{R}(a_{(1)}, b)\,.\,\mathscr{R}(a_{(2)}, c)$$

$$\mathscr{R}(b_{(1)}, a_{(1)})a_{(2)}\,.\,b_{(2)} = b_{(1)}\,.\,a_{(1)}\mathscr{R}(b_{(2)}, a_{(2)})$$

$$\mathscr{R}^{-1}(a_{(1)}, b_{(1)})\,.\,\mathscr{R}(a_{(2)}, b_{(2)}) = \epsilon(a)\,.\,\epsilon(b) = \mathscr{R}(a_{(1)}, b_{(1)})\,.\,\mathscr{R}^{-1}(a_{(2)}, b_{(2)})$$

$$(a, b) \mapsto \mathscr{R}(a_{(1)}, b_{(1)})\,.\,(b_{(2)}, a_{(2)})\,.\,\mathscr{R}^{-1}(a_{(3)}, b_{(3)})$$

Lu-Yan-Zhu Theory: [LYZ1], [LYZ2]

- Classify finite-dimensional Hopf algebras/**C** with positive basis
- Their proofs work for Hopf algebras in Rel
- Any Hopf algebra on a set G in Rel = Hopf algebra structure on a free lattice $\mathscr{P}(G)$ in SupLat
- [LYZ1] Any such Hopf algebra is the bicrossproduct of a group algebra $\mathscr{P}(G_+)$ and a function algebra $\mathscr{P}(G_-)$

$$\epsilon(g) = \begin{cases} 1 & \text{iff } g \in G_- \\ \varnothing & \text{otherwise} \end{cases}$$

- [LYZ2] A CQ structure on $\mathscr{P}(G_+ . G_-)$ corresponds to a pair of group morphisms $\eta, \xi : G_- \to G_+$ satisfying ....

$$(g_-, h_-) \longmapsto \left({}^{\eta(g_-)}h_-, g_-^{\xi(h_-)}\right)$$

- Any group with braiding operator give rise to such data trivially

background

Question: Can skew braces be viewed as Hopf algebras in a reasonable category?

Answer: *Yes!*

a good categorical interpretation should:

- Explain aspects of skew braces in terms of Hopf algebras
- Categorical FRT should recover the universal skew brace
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Explain aspects of skew braces in terms of Hopf algebras

Theorem 4.3 of [G1]

If skew brace $(B, ., \star)$ arises as the remnant of $(H, \mathscr{R})$, then

$$\star = \pi \text{"}\textit{transmuted product}\text{"}(\iota \otimes \iota)$$

Transmutation Theory [Maj]

Given any co-quasitriangular Hopf algebra $(H, \mathscr{R})$, we have a new product and antipode

$$a \star b = \mathscr{R}\left(S(a_{(2)}) \otimes b_{(1)}S(b_{(3)})\right) a_{(1)} \,.\, b_{(2)}$$

$$S^{\star}(a) = \mathscr{R}\left(a_{(1)} \otimes S(a_{(4)})S^{2}(a_{(2)})\right) S(a_{(3)})$$

making H a braided Hopf algebra in the category of left H-comodules.

Transmutation in the category of *right* comodules would give skew *right* braces

Categorical FRT should recover the universal skew brace

A. *Braided object in arbitrary monoidal category:*

B. *Functor interpretation*

C. *Dualizable braided object*

D. *Tannaka-Krein reconstruction*

$$H_\omega := \int^{a \in \mathscr{B}} \omega(a) \otimes \omega(a)^\vee$$

A. *Braided object in arbitrary monoidal category:*

B. *Functor interpretation*

C. *Dualizable braided object*

D. *Tannaka-Krein reconstruction*

$$H_\omega := \int^{a \in \mathscr{B}} \omega(a) \otimes \omega(a)^\vee$$

H_ω is the SupLat-algebra generated by elements $(x, y)_1$ and $(x, y)_2$ corresponding to $x, y \in X$ and imposing relations

$$\vee_{a\in X} \{(x,a)_1 . (x,a)_2\} = \{1\} = \vee_{a\in X} \{(a,x)_2 . (a,x)_1\}$$

$$(x,a)_1 . (y,a)_2 = \varnothing = (a,x)_2 . (a,y)_1 \qquad x \neq y$$

$$(x,y)_1 . (a,b)_1 = (\sigma_x(a), \sigma_y(b))_1 . (\gamma_a(x), \gamma_b(y))_1$$

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$$(x,y)_1 \,.\, (a,b)_1 = (\sigma_x(a), \sigma_y(b))_1 \,.\, (\gamma_a(x), \gamma_b(y))_1$$

$\epsilon((\underline{x}, \underline{y})_{\underline{i}}) = 1$ if and only if $x_{i_j} = y_{i_j}$ for all $1 \leq j \leq n$

$$\Delta((\underline{x}, \underline{y})_{\underline{i}}) = \vee\left\{\left((\underline{x}, \underline{l})_{\underline{i}}, (\underline{l}, \underline{y})_{\underline{i}}\right) \mid \forall\, l \in X^n\right\}$$

Apply Hopf algebraic techniques to skew braces

Drinfeld co-twists

A co-twist on a CQ Hopf algebra $(H, \mathscr{R})$ consists of $\mathscr{F} : H \otimes H \to \mathscr{P}(1)$ satisfying

$$\mathscr{F}^{-1}(a_{(1)}, b_{(1)}) \,.\, \mathscr{F}(a_{(2)}, b_{(2)}) = \epsilon(a) \,.\, \epsilon(b) = \mathscr{F}(a_{(1)}, b_{(1)}) \,.\, \mathscr{F}^{-1}(a_{(2)}, b_{(2)})$$

$$\mathscr{F}\left(a_{(1)} \,.\, b_{(1)}, c\right) \,.\, \mathscr{F}\left(a_{(2)}, b_{(2)}\right) = \mathscr{F}\left(a, b_{(1)} \,.\, c_{(1)}\right) \,.\, \mathscr{F}\left(b_{(2)}, c_{(2)}\right)$$

$$\mathscr{F}(a,1) = \epsilon(a) = F(1,a)$$

we obtain a new CQHA $(H^{\mathscr{F}}, \mathscr{R}^{\mathscr{F}})$

$$m^{\mathscr{F}}(a,b) = \mathscr{F}^{-1}\left(a_{(1)}, b_{(1)}\right) \,.\, a_{(2)} \,.\, b_{(2)} \,.\, \mathscr{F}\left(a_{(1)}, b_{(1)}\right)$$

$$\mathscr{R}^{\mathscr{F}}(a,b) = \mathscr{F}^{-1}\left(a_{(1)}, b_{(1)}\right) \mathscr{R}(a_{(2)}, b_{(2)}) \,.\, \mathscr{F}\left(b_{(3)}, a_{(3)}\right)$$

Drinfeld Twists on Skew Braces [G2]

Consists of a triple of bijections $F : G^2 \to G^2$ and $\Phi, \Psi : G^3 \to G^3$ satisfying

$$F_{12}\Psi = F_{23}\Phi$$

$$\Psi r_{23} = r_{23}\Psi \qquad \Phi r_{12} = r_{12}\Phi$$

$$F(e,x) = (e,x), \quad F(x,e) = (x,e)$$

$$\Psi(x,y,e) = (x,y,e), \quad \Phi(e,x,y) = (e,x,y)$$

$$m_{23}\Phi = Fm_{23} \qquad m_{12}\Psi = Fm_{12}$$

Any co-twist $\mathscr{F}$ on a co-quasitriangular Hopf algebra $(H, \mathscr{R})$ induces a twist on its remnant skew brace $\mathsf{R}(H)$.

If (F, Φ, Ψ) is a Drinfeld twist on a group (G, m, e) with a braiding operator r,

Then (G, mF^{-1}, e) defines a new group structure G with a braiding operator FrF^{-1}.

Classifying Drinfeld Twists

For any twist $(F, \Phi, \Psi) : (G, ., \star) \to (G, \circ, \star)$ there exists a family of group isomorphisms $\{f_x : (G, \star) \to (G, \star')\}_{x \in G}$ satisfying $f_x(x) = x$ so that

$$F(x, y) = \left(f_{x.y}(x), \sigma^{-1}_{f_{x.y}(x)}\left(f_{x.y}(\sigma_x(y))\right) \right) = \left(f_{x.y}(x), f_{x.y}(x)^{\circ} \circ (x \,.\, y) \right)$$

The study of Drinfeld shows that consequences from SupLat go beyond LYZ theory

Much More left to do:

1) Understand Bachiller's work in terms of comodules

Linear YBE Solutions ← Rep theory — Co-quasitriangular Hopf algebras

Linear YBE Solutions — FRT → Co-quasitriangular Hopf algebras

Set-theoretical YBE Solutions ← Bachiller's result — Skew braces

Set-theoretical YBE Solutions — Universal Group → Skew braces

2) Apply Co-double bosonisation to get new skew braces!

3) Combinatorial knot Invariants = Quantum invariants?

Skew braces ⊆ biquandles

$U_q(\mathfrak{g})$, (co)-quasitriangular Hopf algebras

Thank You for your Attention!

Slides/references available at my website:
https://sites.google.com/view/aghobadimath